ZERO KNOWLEDGE PROOF

WEB3零知識證明 高效密碼運算算法

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課程安排

零知識證明高效密碼運算算法

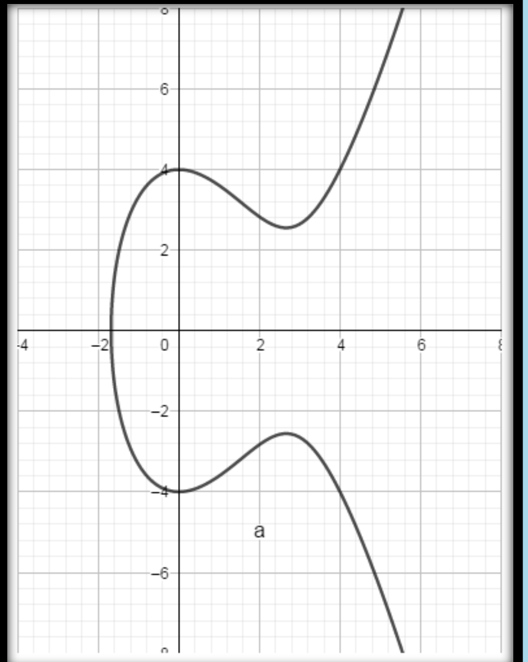
- 快速橢圓曲綫翻倍和加法算法和乘法算法
- NTT 數論變換算法
- MSM 多標量乘法

神奇的橢圓曲綫的發展

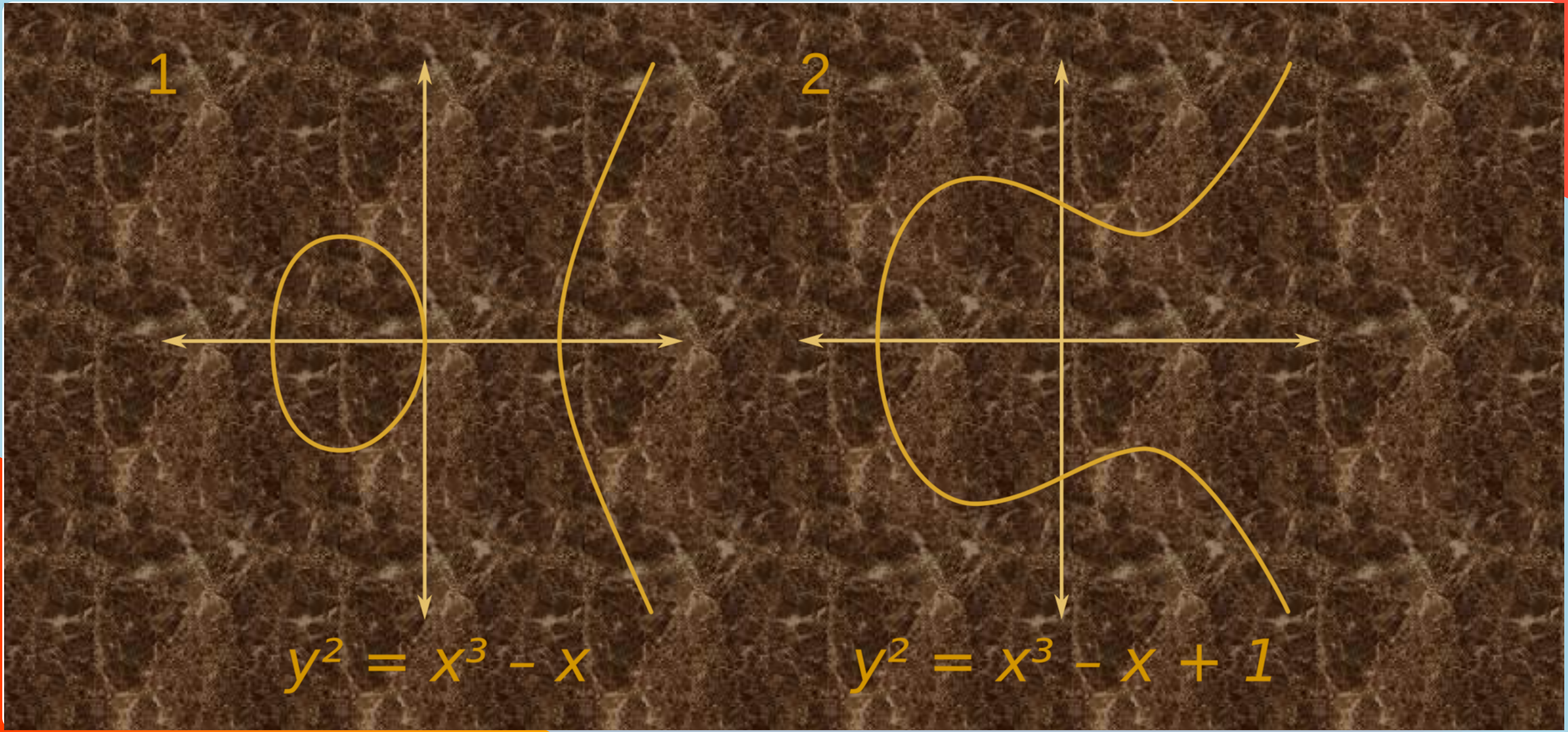
- 椭圆曲线方程来源于椭圆积分，后者来最初来源于计算椭圆周长的问题，有一段时间的历史了，在欧拉时期就开始研究
- 对于椭圆曲线上的点和O点组成的集合，以及集合上定义的二元加法运算，构成一个**Abel**群。单位元是O点，P(x,y)的逆元是P(x,-y)，封闭性，结合性以及交换性也是显然满足的

密码学中普遍采用的是有限域上的椭圆曲线，也即是变元和系数均在有限域中取值的椭圆曲线。使用模素数p的有限域Zp，将模运算引入到椭圆曲线算术中，变量和系数从集合0,1,2,...,p-1中取值而非是在实数上取值

有限域上的椭圆曲线的点和加法运算构成一个有限交换群S



橢圓曲綫大观园

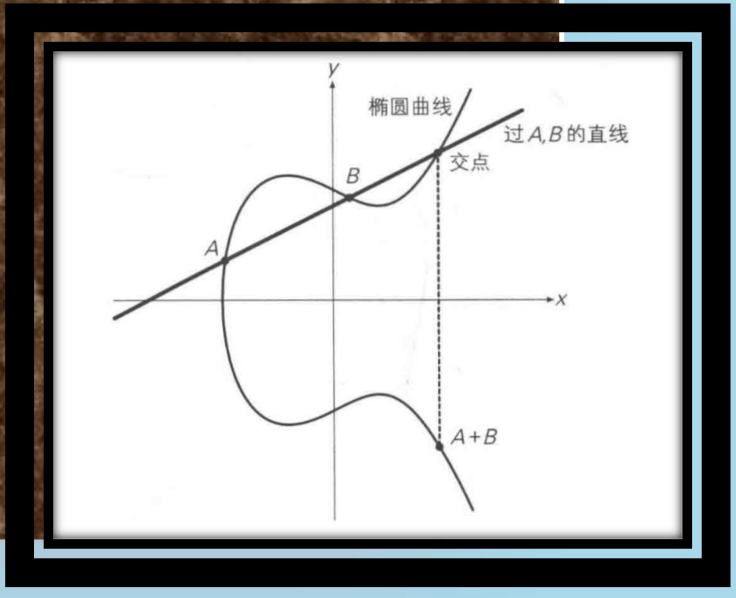


快速橢圓曲綫翻倍和加法算法

- *p a prime #*
- *g a group member of cyclic order p =>* $g*p=1_G$
- *we can think about g as a point on the elliptic curve*
- **We like to calculate [n]g**
- Classical way to compute this

  (((g + g) + g) + g) + g …….
- Another way: repeatedly doubling $2^{\wedge}kg$

  $P_0 = g; P_1 = 2P_0; P_j = 2P_{j-1}$ 。。。。。。



快速橢圓曲綫翻倍和加法算法

$Q = g$

For $i = 1$ to t

$Q = 2Q$

if $e_i = 1$ $R = R + Q$

Return R (ng)

This is the so-called Squared and Plus operation

DAA ②

First, we write out the binary expansion of n,

$\lambda - 1 = t$

$$n = e_0 + e_1 \cdot 2 + e_2 \cdot 2^2 + \cdots + e_{\lambda-1} 2^{\lambda-1}$$

so $e_0, \dots, e_{\lambda-1}$ is the binary representation of n

output accumulator

op identity, base pt, pt at ∞, $(0,1,0)$

Set our double accumulator $Q = P$, $R = \begin{cases} O & e_0 = 0 \\ P & e_0 = 1 \end{cases}$

快速橢圓曲綫乘法群算法

For a multiplicative group (any abelian group, not even cyclic) G, $g \in G$,

if $n = \sum e_i 2^i$

compute $g^n = \prod (g^{2^i})^{e_i}$

in $\le \log_2 n$ squarings

and $\le \log_2 n$ multiplications

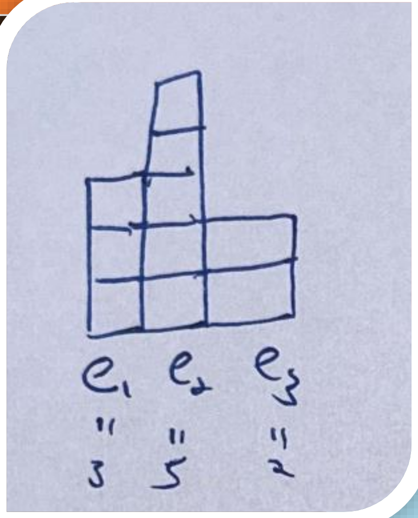
If n is a random #, it will be expected half of the bits are ones and half of the bits are zeroes. So, $\log_2 n$ squares and ½ $\log_2 n$ additions

橢圓曲綫乘法快速高效算法

- Start with n = 3

$G = g_1^{e_1} g_2^{e_2} g_3^{e_3}$ G a group

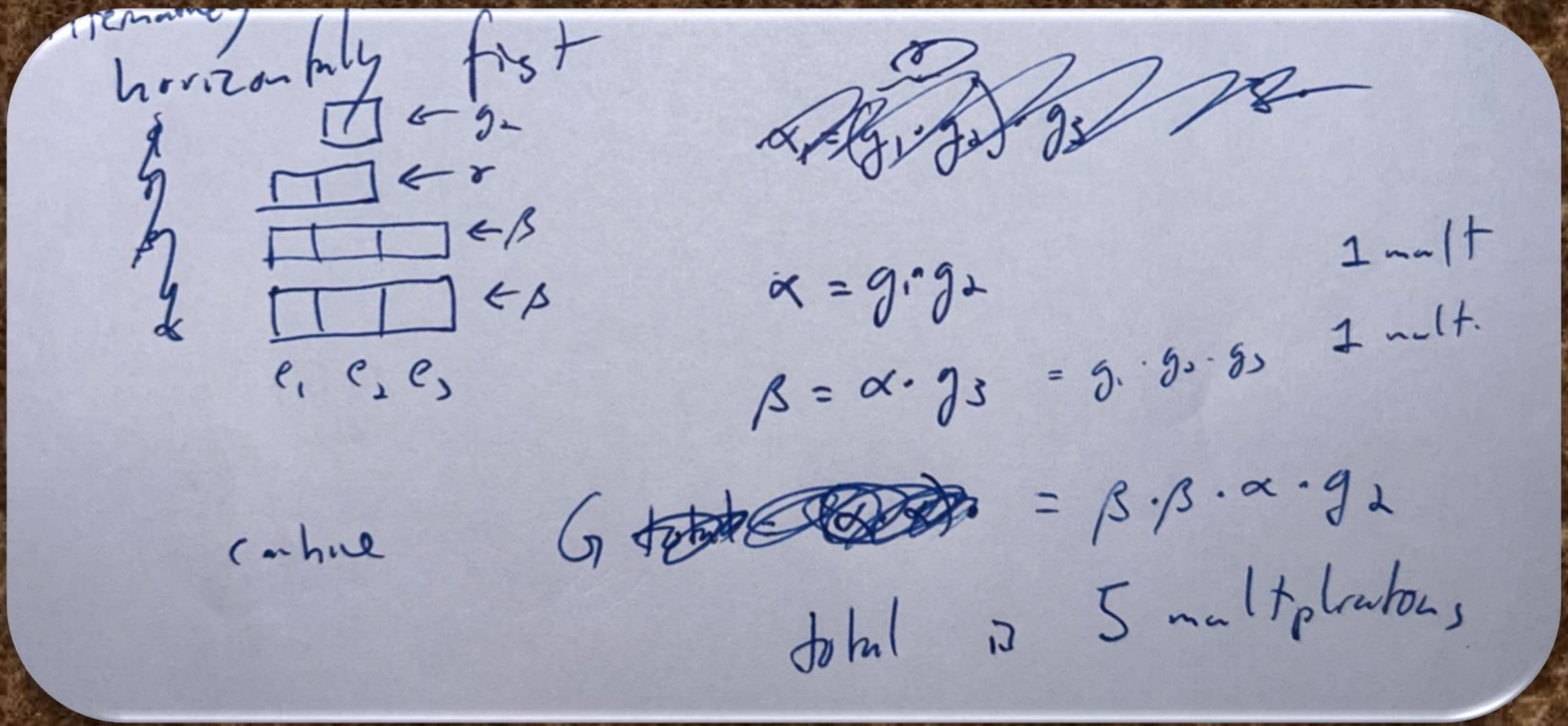
- If $e_i = \{0, 1\}$ then **G** is the multi-product



- $G = g_1 * g_1 * g_1\ g_2 * g_2 * g_2\ g_2 * g_2 * g_3 * g_3$
- Model the operation cost as either the multiplication or squaring
- We calculate this example, **2 + 3 + 1 + 2 = 8**

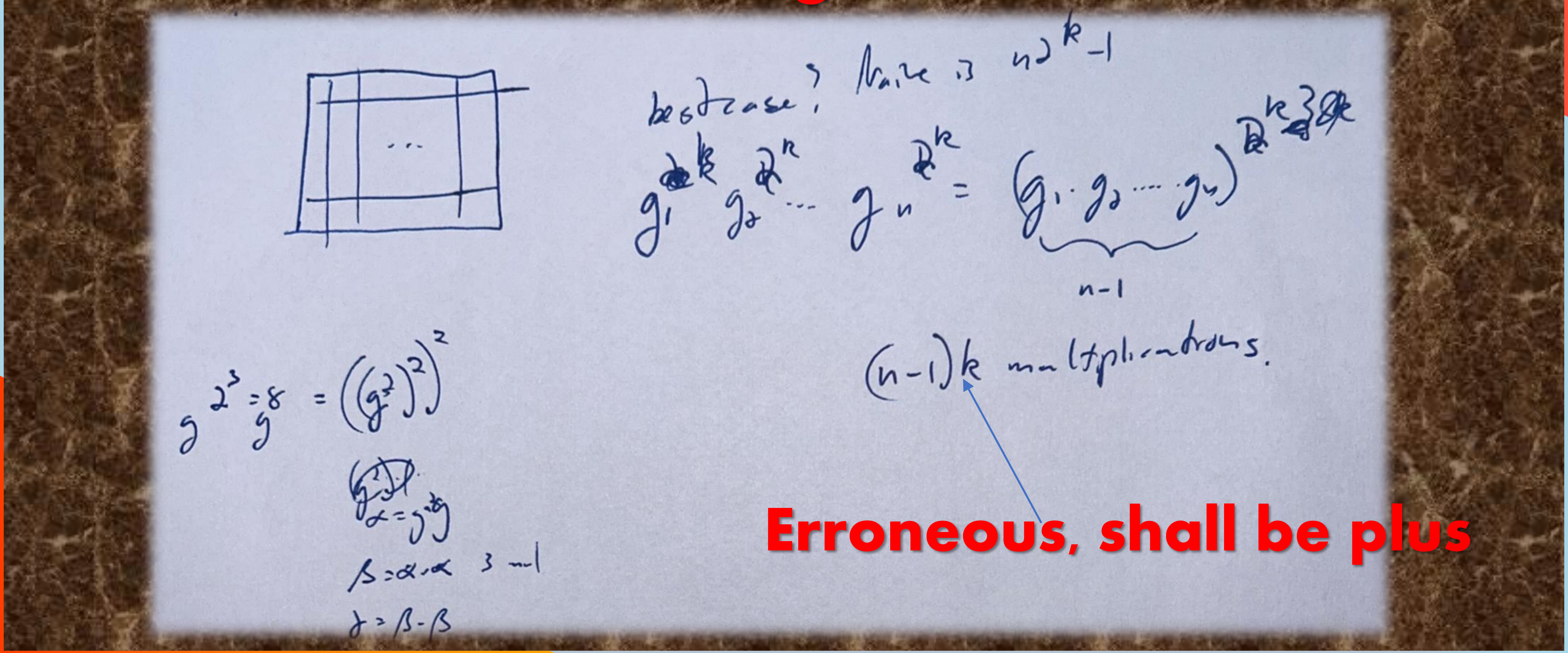
橢圓曲綫乘法快速高效算法

- **Another way, Abelian sum method or Lebesgue integration method**



橢圓曲綫乘法快速高效算法

- Now we move to the general situation



Erroneous, shall be plus

椭圆曲线乘法快速高效算法

Now we'll use some information about the group, Assume G is a cyclic group of order p, where p is a $\approx \lambda$-bit prime, 254 bit so $\lambda = 256$,

$$G = \prod_{i=0}^{N-1} g_i^{e_i} \quad \text{msm}$$

Assume $\lambda \geq N$ and decompose λ into s legs, $\lambda =$ s.t, say $s = 4$ legs and $t = 64$ of size t each w/ $N = 16$. (turns out $s \approx \sqrt{\frac{\lambda}{N}}$ and $t \approx \sqrt{\lambda N}$ is optimal and convenient)

橢圓曲綫乘法快速高效算法

Let $e_i = \sum_{\ell=0}^{\lambda-1} \overbrace{e_{i,\ell}}^{\in\{0,1\}} 2^{\ell}$ so the $e_{i,\ell}$ are the binary digits of e_i lowest-order to highest. little-endian

splitting λ into legs

$$e_i = \sum_{j=0}^{s-1} \sum_{k=0}^{t-1} 2^{j+sk} \underbrace{e_{i,j+sk}}_{0 \text{ or } 1}$$

Then $g_i^{e_i} = \prod_{\ell=0}^{\lambda-1} g_i^{2^{\ell} e_{i,\ell}} \underset{\lambda=st}{=} \prod_{j=0}^{s-1} \prod_{k=0}^{t-1} g_i^{2^{j+sk} \underbrace{e_{i,j+sk}}_{0 \text{ or } 1}}$

橢圓曲綫乘法快速高效算法

so $$G = \prod_{i=0}^{N-1} g_i^{e_i} = \prod_{i=0}^{N-1} \left(\prod_{j=0}^{s-1} \prod_{k=0}^{t-1} g_i^{2^{j+sk} e_{i,j+sk}} \right)$$

and rearranging the product,

$$G = \prod_{k=0}^{t-1} \Bigg(\underbrace{\prod_{i=0}^{N-1} \prod_{j=0}^{s-1} g_i^{2^j e_{i,j+sk}}}_{:= G_k'} \Bigg)^{2^{sk}}$$

repeated squaring

橢圓曲綫乘法快速高效算法

- So, now we only need to figure out how to handle those multiplication products inside the parentheses G_k'
- Learning from the early part, we would like to utilize the squared terms such as

$$g_i, g_i^2, g_i^4, \ldots g_i^{2^j}, \ldots g_i^{2^{s-1}}$$

- The total cost is N*S = N * $\sqrt{\lambda/N} = \sqrt{\lambda N}$

橢圓曲綫乘法快速高效算法

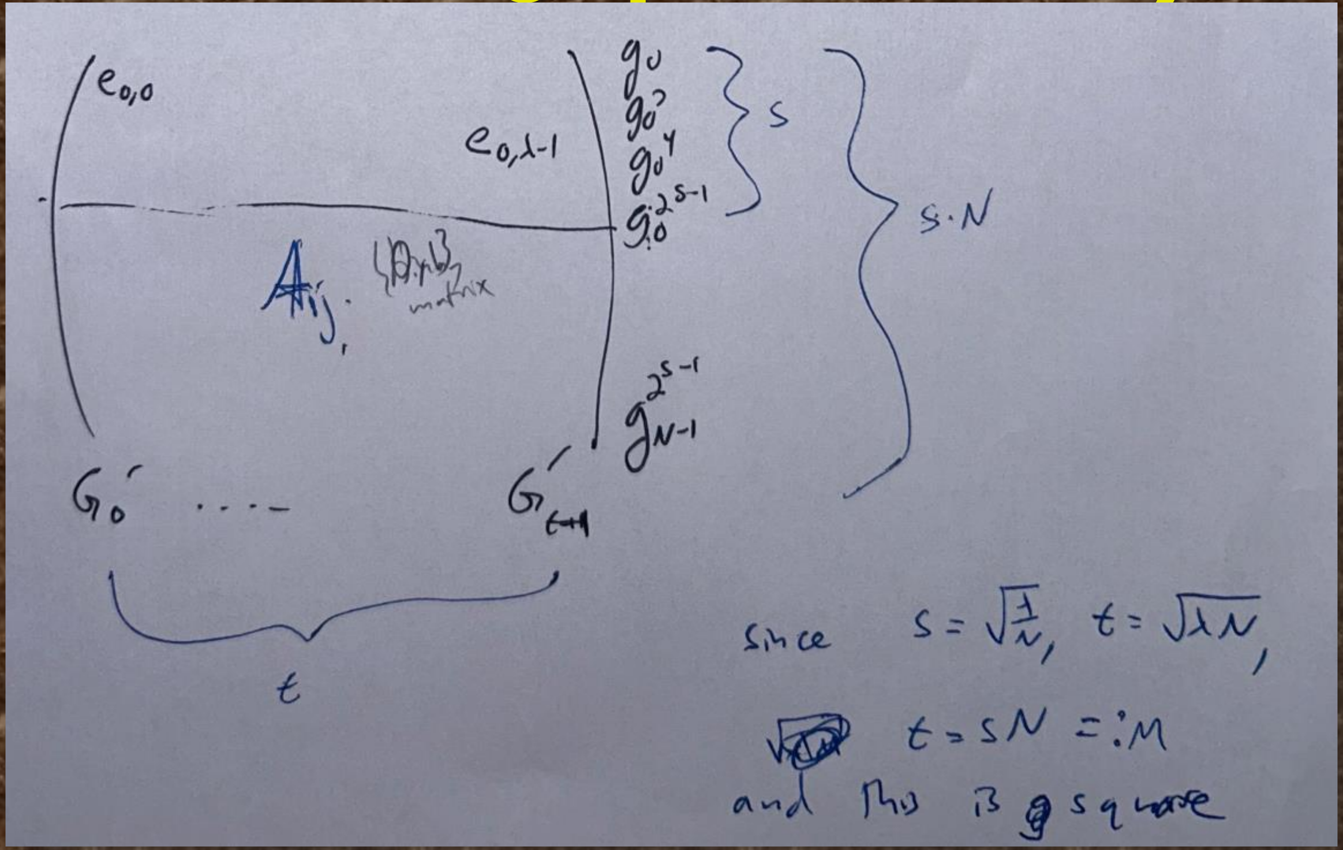
- For such inputs, each inner product term

$$\prod_{i=0}^{N-1} \prod_{j=0}^{S-1} g_i^{\alpha^j e_{i,j+sk}}$$

will be a multiplication of $g_i^{\alpha^j}$

橢圓曲綫乘法快速高效算法

- We will form the following sparse binary square matrix



橢圓曲綫乘法快速高效算法

The first matrix is as follows

$$\begin{pmatrix}
e_{0,0} & e_{0,s} & e_{0,2s} & & e_{0,(t-3)s} & e_{0,(t-2)s} & e_{0,(t-1)s} \\
e_{0,1} & e_{0,s+1} & e_{0,2s+1} & \cdots & e_{0,(t-3)s+1} & e_{0,(t-2)s+1} & e_{0,(t-1)s+1} \\
e_{0,2} & e_{0,s+2} & e_{0,2s+2} & & e_{0,(t-3)s+2} & e_{0,(t-2)s+2} & e_{0,(t-1)s+2} \\
 & \vdots & & \ddots & & \vdots & \\
e_{0,s-3} & e_{0,2s-3} & e_{0,3s-3} & & e_{0,(t-2)s-3} & e_{0,(t-1)s-3} & e_{0,ts-3} \\
e_{0,s-2} & e_{0,2s-2} & e_{0,3s-2} & \cdots & e_{0,(t-2)s-2} & e_{0,(t-1)s-2} & e_{0,ts-2} \\
e_{0,s-1} & e_{0,2s-1} & e_{0,3s-1} & & e_{0,(t-2)s-1} & e_{0,(t-1)s-1} & e_{0,ts-1}
\end{pmatrix}$$

橢圓曲綫乘法快速高效算法

The k-th block matrix is as follows

$$
\begin{pmatrix}
e_{k,0} & e_{k,s} & e_{k,2s} & & e_{k,(t-3)s} & e_{k,(t-2)s} & e_{k,(t-1)s} \\
e_{k,1} & e_{k,s+1} & e_{k,2s+1} & \cdots & e_{k,(t-3)s+1} & e_{k,(t-2)s+1} & e_{k,(t-1)s+1} \\
e_{k,2} & e_{k,s+2} & e_{k,2s+2} & & e_{k,(t-3)s+2} & e_{k,(t-2)s+2} & e_{k,(t-1)s+2} \\
 & \vdots & & \ddots & & \vdots & \\
e_{k,s-3} & e_{k,2s-3} & e_{k,3s-3} & & e_{k,(t-2)s-3} & e_{k,(t-1)s-3} & e_{k,ts-3} \\
e_{k,s-2} & e_{k,2s-2} & e_{k,3s-2} & \cdots & e_{k,(t-2)s-2} & e_{k,(t-1)s-2} & e_{k,ts-2} \\
e_{k,s-1} & e_{k,2s-1} & e_{k,3s-1} & & e_{k,(t-2)s-1} & e_{k,(t-1)s-1} & e_{k,ts-1}
\end{pmatrix}
$$

$$G_k' = \prod_{i=0}^{N-1}\prod_{j=0}^{s-1} g_i^{2^j e_{i,j+sk}}$$

橢圓曲綫乘法快速高效算法

- The expression $\prod_{j=0}^{s-1} g_i^{2^j e_{i,j+sk}}$ is obtained by

橢圓曲綫乘法快速高效算法

- A simple strategy for evaluating these expressions is to pick a partition of the vertical vector $S_0, \dots S_{M/b-1}$ of at most b elements
- For each S_i, we compute all possible multiplications, denoted as T_i
- For example, for b = 3, then, $S_0 = \{ h_0, h_1, h_2 \}$

$T_0 = \{ h_0 , h_1 , h_2 , h_0h_1, h_0h_2 , h_1h_2 , h_0h_1h_2 \}$

橢圓曲綫乘法快速高效算法

- G_k' is a product of at most one element from each T_i

Analysis: each S_i has b elts, 2^b gp ops to compute all possible multiproducts / contributions.

M/b sets, so this precompute is $\frac{2^b M}{b}$ gp ops.

Given the T_i, Each H_i needs at most one elt from each set,

so $\frac{M}{b}$ gp ops

~~M His be~~

There are M of the H_is, so $\frac{M^2}{b}$ given the T_i

橢圓曲綫乘法快速高效算法

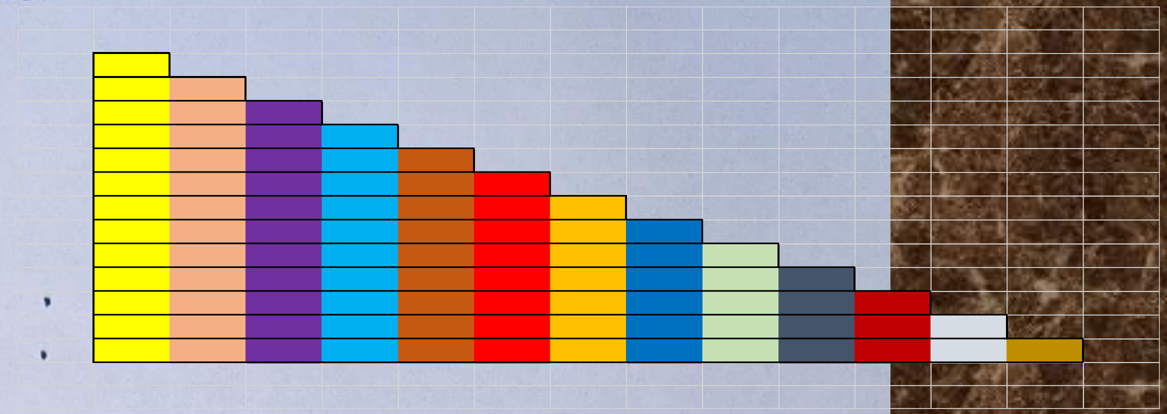
Finally we recombine inputs

$$G = \prod_{k=0}^{t-1} G_k'^{\,2^{sk}}$$

using $st = \lambda$ squarings:

square G'_{t-1} s times,

multiply by G'_{t-2}, square that s times, ...



橢圓曲綫乘法快速高效算法

$M = \sqrt{\lambda N} \quad : \quad t = sN$

$$\lambda \quad + \quad M \quad + \quad 2^b \frac{M}{b} \quad + \quad \frac{M^2}{b}$$

squarings in $\prod_{k=0}^{t-1} G_k^{2^{sk}}$

multiplications in

precompute contributions T_i

compute $H_k = G_k$ given contributions T_i

橢圓曲綫乘法快速高效算法

- For one of the optimal choices of b, we have

$$b = \log M - \log\log M,$$

- Plug this choice of b into the above calculation

$$\lambda + M + \frac{M^2}{(\log M - \log\log M)(\log M)} + \frac{M^2}{\log M - \log\log M}$$

$$= \lambda + (1+o(1))\frac{M^2}{\log M}$$

椭圆曲线乘法快速高效算法

- We put everything together to obtain the final optimal cost result

$M = \sqrt{\lambda N}$ so cost is

$$\lambda + (1+o(1))\frac{2\lambda N}{\log \lambda N}$$

why doesn't it depend on the degree bound?

λ is the degree bound

Maybe if actual exponents are $e \ll p$, there is a better way.

FFT&DFT高效算法

- We are goanna deal with polynomials

The analogy here (can be made precise!) is that
a polynomial can be given in several forms
as well, including

1. as coefficients   expanded $3 + 4x + 7x^2 + 5x^3 + \dots$

   or

   $3x_1x_2x_{10} + 17x_3x_7x_{30} + \dots$

or 2. evaluated at points (need degree + 1 points for
a univariate polynomial, # of monomials of degree $\leq n$ o/w
w/ caveats

FFT&DFT高效算法

Thm if $x_1,\dots,x_p \in \mathbb{F}^m$ distinct points in $\mathbb{F}^m$, TFAE

(1) given $y_1,\dots,y_p \in \mathbb{F}$

$\exists!\, f \in F[\vec{x}]$ of deg $\le n$

s.t. $f(x_i) = y_i \quad \forall\, 1 \le i \le p$

(2) Sample $p \times p$ matrix

$M = \left(x_i^{I_j} \right)$ is invertible

Vandermonde w/ $m=1$.

FFT&DFT高效算法

$$
\begin{array}{rl}
& a_0 + a_1 x + a_2 x^2 + a_3 x^3 = f \\
+ & \underline{b_0 + b_1 x + b_2 x^2 + b_3 x^3} = g \\
& (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + (a_3+b_3)x^3
\end{array}
$$

deg + 1 additions

<u>In evaluation form</u>

$(f+g)(1) = f(1) + g(1)$

$(f+g)(2) = f(2) + g(2)$ enough to determine f+g in eval form

$(f+g)(3) = f(3) + g(3)$

$(f+g)(4) = f(4) + g(4)$ ← ~~deg + deg~~ degree + 1

FFT&DFT高效算法

Polynomial multiplication

- ZK-snark constructions use polynomial multiplications
- The polynomials involved are univariate and high degree.

Let's say we have coeff-form poly f, g & want coeff-form-poly fg

$\deg f = 2$

$$f(x) = a_0 + a_1 x + a_2 x^2$$

$$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

naively we need $(\deg f + 1)(\deg g + 1)$ (field) multiplications

$$f \cdot g = (a_0 + a_1 x + a_2 x^2)(b_0 + b_1 x + b_2 x^2 + b_3 x^3)$$

FFT&DFT高效算法

$$f(x) = a_0 + a_1 x + a_2 x^2$$

$$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

naively we need $(\deg f + 1)(\deg g + 1)$ multiplications (field)

$$f \cdot g = (a_0 + a_1 x + a_2 x^2)(b_0 + b_1 x + b_2 x^2 + b_3 x^3)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)$$

$$(f \cdot g)_\ell = \sum_{j+k=\ell} a_j b_k = \sum_{j=0}^{\ell} a_j b_{\ell-j}$$

"$O(n^2)$"

But $(f \cdot g)(x) := f(x) \cdot g(x)$ (the definition above is fact)

FFT&DFT高效算法

But $(f \cdot g)(x) := f(x) \cdot g(x)$ (the definition above in fact)

so in evaluation form, we only need deg + 1 multiplications " ~~nearly linear time~~ " $O(n)$

so the idea is to multiply polynomials in coefficient form, change them to evaluation form, multiply, change back.

This is good if the changes are cheap, FFT makes it $O(n \log n)$

FFT&DFT高效算法

- N-th unit roots $\omega^n = 1$
- Prime n-th unit root, if for any natural # q < n $\omega^q \mathrel{!=} 1$

So the idea a is that the coefficient ring of the polynomials (here $\mathbb{F}_p$, a field) should contain certain roots of unity, w

we p use, instead of evaluations $f(1), f(2) \ldots,$ evaluations $f(w^1), f(w^2), \ldots,$ and this makes interpolating f from its evaluations efficient. Thus eval form $\to$ coeff form will be efficient.

FFT&DFT高效算法

NTT ③

Ex $\quad x^{8-1}=1$ so every nonzero elt is a root of unity $(x^8=x)$.

An $n\in\mathbb{Z}_{>0}$ is a primitive root of unity if $x^n=1$ but $x^m\neq 1$ for any $m<n$.

If a is an n^{th} pr.o.u,

$\mathbb{F}$ has all n roots of unity $1, a, a^2, \dots a^{n-1}$.

$\mathbb{F}_q$ contains n^{th} primitive root of

FFT&DFT高效算法

So if $f = \sum_{j=0}^{n-1} f_j x^j$ is a polynomial of n-dim

coeff vector $f_0, \dots, f_{n-1}$,

we have the map "evaluate at powers of ω"

$DFT_\omega : \mathbb{F}_q^n \to \mathbb{F}_q^n$

$f \mapsto (f(1), f(\omega), f(\omega^2), \dots, f(\omega^{n-1}))$

this is the "discrete fourier transform" is an $\mathbb{F}$-linear map

to convolution

FFT&DFT高效算法

We have to adjust the notion of multiplication,
basically the same but degree is mod n

$$f = f_0 + f_1 x + \cdots + f_{n-1} x^{n-1}$$

$$g = g_0 + g_1 x + \cdots + g_{n-1} x^{n-1}$$

convolved

$$h = f *_n g = \sum_{l=0}^{n-1} h_l x^l$$

$0 \le l \le n$

$$\text{w/ } h_l = \sum_{j+k \equiv l \bmod n} f_j g_k = \sum_{j=0}^{n-1} f_j g_{l-j}$$

FFT&DFT高效算法

$$\underbrace{2x^6+3x^5+x^4}_{\text{upper}} + \underbrace{3x^3+3x^2+x+1}_{\text{lower}} \quad \bmod x^4-1$$

$$= (2x^2+3x+1)(x^4-1) + \text{lower} + \underbrace{(2x^2+3x+1)}_{\text{add back}}$$

$$= \underbrace{3x^3+3x^2+x+1}_{\text{lower}} + \underbrace{(2x^2+3x+1)}_{\text{upper}} \quad \bmod (x^4-1)$$

$$= 3x^3+5x^2+4x+2 \quad \bmod (x^4-1)$$

FFT&DFT高效算法

$*_n$ is cyclic convolution.

Equivalent to polynomial multiplication in $\mathbb{F}_q[x] / \langle x^n - 1 \rangle$

$$f * g \equiv fg \mod x^n - 1$$

If $\deg(fg) < n$, then $f * g = fg \mod (x^n - 1)$

implies $fg = f * g$

so need 2^kth root funct

w/ $2^{k-1} < 2n \leq 2^k$

FFT&DFT高效算法

So we want to find a $kn \log n$ and show DFT & inverse is $O(n \log n)$

(poly mult will be $18 n \log n + O(n)$)

Vandermonde matrix

$$V_\omega = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^{2(n-1)} \\ \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} \cdots & \omega^{(n-1)^2} \end{pmatrix} = (\omega^{jk})$$

$0 \le j, k < n$

is the matrix of the linear multipoint evaluation map coeff → evals at $1, \omega, \ldots$

FFT&DFT高效算法

V_ω is invertible, $(V_\omega)^{-1} = \frac{1}{n} V_{\omega^{-1}}$ so everything works both ways

But applying V_ω can be done faster than taking n, n-size dot products.

FFT&DFT高效算法

The idea is as follows

n even (usually $n=2^k$ for some k)

w a primitive n^{th} root of unity

f deg $< n$

To evaluate f at $1, w, w^2, \dots, w^{n-1}$,

we divide f by $x^{n/2}-1$

and $x^{n/2}+1$ with remainder:

obtaining quotients

FFT&DFT高效算法

To evaluate f at $1, \omega, \omega^2, \ldots, \omega^{n-1}$,

we divide f by $x^{n/2} - 1$

and $x^{n/2} + 1$ with remainder:

obtaining quotients + remainders

write $f = q_0(x^{n/2} - 1) + r_0 = q_1(x^{n/2} + 1) + r_1$

w/ $q_0, r_0, q_1, r_1 \in \mathbb{F}[x]$, $\deg < \frac{n}{2}$

we only need the remainders, and (Ex),

- we can get r_0 by adding the upper $\frac{n}{2}$ coeff of f to the lower $\frac{n}{2}$ coeff

FFT&DFT高效算法

NTT 12

Then plug in a power of w

$$f(w^{2l}) = q_0(w^{2l})(w^{2l}-1) + r_0(w^{2l}) = r_0(w^{2l})$$

and $$f(w^{2l+1}) = q_1(w^{2l+1})(w^{nl}w^{n/2}+1) + r_1(w^{2l+1}) = r_1(w^{2l+1})$$

for $0 \le l < \frac{n}{2}$. (note $w^{nl} = 1$

$w^{n/2} = -1$

since

$0 = w^n - 1 = (w^{n/2}-1)(w^{n/2}+1)$)

Erroneous

FFT&DFT高效算法

So we have all the eval pts

split into even & odd powers of deg ≤ n f

$$f(\omega^{2\ell}) = r_0(\omega^{2\ell})$$

$$f(\omega^{2\ell+1}) = r_1(\omega^{2\ell+1}) = r_1(\omega(\omega^{2\ell}))$$

but, r_0, r_1 have degree $\leq \frac{n}{2}$

- ω^2 is a primitive $(\frac{n}{2})^{th}$ root of unity

So we recurse.

FFT&DFT高效算法

FFT Alg

Input: $n = 2^k$ $\quad f = \sum_{j=0}^{n-1} f_j x^j$

$\omega, \omega^2, \ldots, \omega^{n-1}$ powers of a primitive nth root of unity

Output: $DFT_\omega(f) = (f(1), f(\omega), \ldots, f(\omega^{n-1})) \in \mathbb{F}_q^n$

1. If $n = 1$ return f_0
2. $r_0 \leftarrow \sum_{0 \le j < \frac{n}{2}} (f_j + f_{j+\frac{n}{2}}) x^j$

$r_1^* \leftarrow \sum_{0 \le j < \frac{n}{2}} (f_j - f_{j+\frac{n}{2}}) \omega^j x^j$ $\quad r_1^* = r_1(\omega x)$

3. recurse to evaluate r_0 and r_1^* at powers of ω^2
4. return
   $r_0(1), r_1^*(1), r_0(\omega^2), r_1^*(\omega^2), \ldots, r_0(\omega^{n-2}), r_1^*(\omega^{n-2})$

$n \log n$ additions in $\mathbb{F}$

$\frac{n}{2} \log n$ multiplications by powers of ω

Total $\frac{3}{2} n \log$ field ops.

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Thank You